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On Motion of Robot End-Effector using the Curvature Theory of Timelike Ruled Surfaces with Timelike Directrix

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ABSTRACT

In this paper, we deal with the differential geometric properties of robot end-effector's motion by using the curvature theory of timelike ruled surfaces with timelike directrix.

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Keywords: Robot end-effector, Ruled surface, Timelike surface, Darboux vector.

1. INTRODUCTION

The methods of robot trajectory control currently used are based on PTP point to point and CP continuous path methods. These methods are basically interpolation techniques and therefore, are approximations of the real path trajectory (see Paul (1979)). In such cases, when a precise trajectory is needed or we need to trace a free formed or analytical surface accurately, the precision is only proportional to the number of intermediate data points for teach-playback or offline programming.

For accurate robot trajectory, the most important aspect is the continuous representation of orientation whereas the position representation is relatively easy. There are methods such as homogeneous transformation, Quaternion and Euler Angle representation to describe the orientation of a body in a three-dimensional space (see Ryuh and Pennock (1988)). These methods are easy in concept but have high redundancy in parameters and are discrete representation in nature rather than continuous. Therefore, a method based on the curvature theory of a ruled surface has been proposed as an alternative in the dissertation of Ryuh (1989).

The ruled surfaces are swept out by a straight line moving along a curve. McCarthy and Roth (1981) have studied in kinematics by many investigators based primarily on line geometry. Since a ruled surface is a special case of a smooth surface, its differential geometry can be developed by using traditional techniques of vector calculus. McCarthy (1987) and McCarthy and Roth (1981) used this approach to obtain a scalar curvature theory of line trajectories for spatial kinematics. Wang *et al.* (1997) set up curvature theory in kinematic geometry of mechanism which is in the form and content from plane to space motion.

The robot end-effector motion may also be completely described by the ruled surface and the spin angle. The positional variation and the angular variation of the rigid body are determined by the curvature theory of a ruled surface in Ryuh and Pennock (1988). Ryuh *et al.* (2006) developed dual curvature theory of the ruled surface and applied this theory into the robot trajectory planning.

Ruled surfaces in Minkowski 3-space have been studied in a lot of fields (see Ogrenmis *et al.* (2006)). Coken *et al.* (2008) investigated parallel timelike ruled surfaces with timelike rulings. More information about timelike ruled surfaces in Minkowski 3-space may also be found in Turgut and Hacısalihoglu (1997) and (1998). Recently Ekici *et al.* (2008) have studied the robot end-effector motion for timelike ruled surfaces with timelike ruling.

2. PRELIMINARIES

Let $\mathbb{R}_1^3$ be a Minkowski 3-space with Lorentzian metric $ds^2 = dx_1^2 + dx_2^2 - dx_3^2$. The norm of $x \in \mathbb{R}_1^3$ is denoted by $\|x\| = \sqrt{|< x, x >|}$ where $< \, , \, >$ is the induced inner product in $\mathbb{R}_1^3$.

We say that Lorentzian vector x is spacelike, lightlike or timelike if $<x,x>>0$ and $x=0$, $<x,x>=0$, $<x,x><0$, respectively. A smooth regular curve $\alpha: I \subset \mathbb{R} \to \mathbb{R}_1^3$ is said to be a timelike, spacelike or lightlike curve if the velocity vector $\alpha'(s)$ is a timelike, spacelike, or lightlike vector, respectively (see O'neill (1983)).

For any $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3) \in \mathbb{R}_1^3$, the Lorentzian vector product of x and y is defined as

$$x \wedge y = (x_2y_3 - x_3y_2,\ x_1y_3 - x_3y_1,\ x_2y_1 - x_1y_2). \tag{1}$$

$$e_1 \wedge e_2 = -e_3 \quad e_3 \wedge e_1 = -e_2 \quad e_2 \wedge e_3 = e_1. \tag{2}$$

where $\{e_1, e_2, e_3\}$ are the base of the space $\mathbb{R}_1^3$.

A surface in $\mathbb{R}_1^3$ is called a timelike surface if the induced metric on the surface is a Lorentz metric, that is, the normal on the surface is a spacelike vector. A timelike ruled surface is obtained by a timelike straight line moving a spacelike curve or by a spacelike straight line moving a timelike curve. The timelike ruled surface M is given by the parametrization,

$$\varphi: I \times \mathbb{R} \to \mathbb{R}_1^3, \quad \varphi(s,u) = \alpha(s) + uX(s) \tag{3}$$

in $\mathbb{R}_1^3$ (see Turgut and Hacısalihoglu (1997)).

3. REPRESENTATION OF ROBOT TRAJECTORY BY A RULED SURFACE

The motion of a robot end-effector is referred to as the robot trajectory. The point fixed in the end-effector will be referred to as the Tool Center Point and denoted as TCP (see Ryuh (1988)). Path of a robot may be represented by a tool center point and tool frame of end-effector. In Fig. 1, the tool frame is represented by three mutually perpendicular unit vectors {*O*, *A*, *N*} where *O* is the orientation vector (spacelike), A is the approach vector (timelike), *N* is the normal vector (spacelike). The ruled surface generated by

O is chosen for further analysis without loss of generality. The spin angle ϕ which represents the rotation from the surface binormal vector S_b , about A.

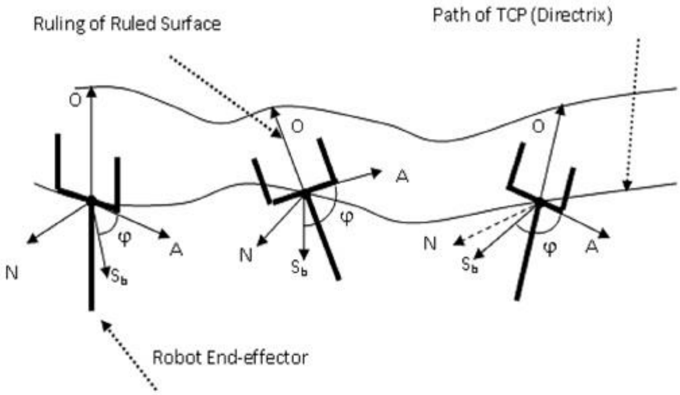

Figure 1: Ruled surface generated by O of tool frame

3.1 Frames of Reference

Let

$$\alpha : I \to \mathbb{R}_1^3$$
$$\mathrm{s} \to \alpha(s)$$

where $\{0\} \subset I$, be a differentiable timelike curve in $\mathbb{R}_1^3$ parameterized by arc-length.

While the robot moves each vector of tool frame in end-effector, it determines its own ruled surface. The path of tool center point is directrix and O is the ruling. As $\alpha(s)$ is a timelike curve and $\vec{R}(s)$ is spacelike straight line, let us take the following timelike ruled surface as

$$X(s,u) = \alpha(s) + u\vec{R}(s) \tag{4}$$

where the space curve $\alpha(s)$ is the specified path of the TCP (called the directrix of the timelike ruled surface), u is a real-valued parameter, and

$\vec{R}(s)$ is the vector generating the timelike ruled surface (called the ruling) (Turgut and Hacısalihoğlu, (1998)).

Note that to determine the orientation of tool frame relative to the timelike ruled surface, we define a surface frame $[\mathrm{O}, S_n, S_b]$ at the TCP as shown in Figs. 1 and 2. S_n is the unit spacelike normal vector and S_b is the unit timelike binormal vector of the timelike ruled surface. They are determined as follows

$$O = \frac{\vec{R}(s)}{\Re}, \quad S_n = \frac{X_s \wedge X_u}{\left\| X_s \wedge X_u \right\|}, \quad S_b = \mathrm{O} \wedge S_n \tag{5}$$

where $\Re = \left\| \vec{R}(s) \right\|$.

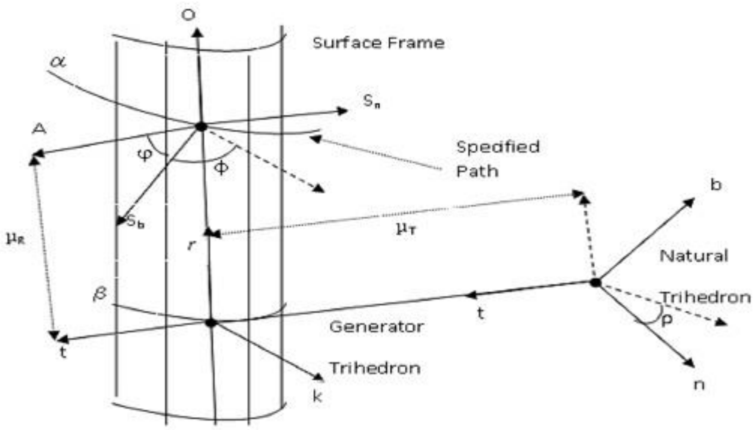

Figure 2: Frames of reference

In Figure 2, the generator trihedron is used to study the positional and angular variation of the timelike ruled surface. r is the unit generator vector (spacelike), t is the unit central normal vector (spacelike) and k is the unit tangent vector (timelike) defined as

$$r = \frac{1}{\Re} \vec{R}(s), \quad t = \vec{R}(s), \quad k = r \wedge t. \tag{6}$$

The origin of the generator trihedron is referred to as the striction point of the timelike ruled surface and the locus is called the striction curve, which is defined by

$$\beta(s) = \alpha(\mathrm{s}) - \mu(\mathrm{s})\vec{R}(\mathrm{s}) \tag{7}$$

where the parameter

$$\mu(s) = \left| < \alpha'(s), \vec{R}'(s) > \right| / \left\| \vec{R}' \right\|$$

indicates the position of the TCP relative to the striction point of the timelike ruled surface.

The first-order angular variation of the generator trihedron may be expressed in the matrix form as

$$\frac{d}{ds}\begin{bmatrix} r \\ t \\ k \end{bmatrix} = \frac{1}{\Re}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -\gamma \\ 0 & -\gamma & 0 \end{bmatrix}\begin{bmatrix} r \\ t \\ k \end{bmatrix} \tag{8}$$

where the geodesic curvature γ is defined as

$$\gamma = < \vec{R}'', \vec{R} \wedge \vec{R}' > . \tag{9}$$

Moreover, the Darboux vector of the generator trihedron is

$$U_r = -\frac{\gamma}{\Re} r - \frac{1}{\Re} k \tag{10}$$

which satisfies $\frac{dr}{ds} = U_r \wedge r,\ \frac{dt}{ds} = U_r \wedge t,\ \frac{dk}{ds} = U_r \wedge k$.

Differentiating Equation (7) with respect to *s*, hence from Equation (8) and Equation (6), we have first order positional variation of the striction point of the timelike ruled surface expressed in the generator trihedron as

$$\beta'(s) = \Gamma r - \Delta k \tag{11}$$

where

$$\Gamma = \frac{1}{\Re} < \alpha'(s), \vec{R}(s) > - \frac{1}{\Re}\mu'(s), \ \Delta = \frac{1}{\Re} < \alpha'(s), \vec{R}(s) \wedge \vec{R}'(s) > . \tag{12}$$

4. CENTRAL NORMAL SURFACE

As the generator trihedron moves along the striction curve, the central normal vector generates another ruled surface called the normalia or the central normal surface which is defined as

$$X_T(s,u) = \beta(s) + ut(s). \tag{13}$$

The natural trihedron of the normalia consists of three orthonormal vectors: t the central normal vector (spacelike), n the principal normal vector (spacelike), and b the binormal vector (timelike) as shown in Figure 2.

Let the hyperbolic angle ρ be between the timelike vectors k and n. Then, we have

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\sinh\rho & 0 & \cosh\rho \\ \cosh\rho & 0 & -\sinh\rho \end{bmatrix} \begin{bmatrix} r \\ t \\ k \end{bmatrix}. \tag{14}$$

Using $t' = \kappa n$ and Equation (8) we have

$$\cosh\rho = -\frac{\gamma}{\Re\kappa}, \ \sinh\rho = \frac{1}{\Re\kappa}, \ \gamma = -\coth\rho. \tag{15}$$

Substituting the Equations (15) into Equation (8), finally we get

$$\frac{d}{ds}\begin{bmatrix} r \\ t \\ k \end{bmatrix} = \frac{1}{\Re}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \coth\rho \\ 0 & \coth\rho & 0 \end{bmatrix} \begin{bmatrix} r \\ t \\ k \end{bmatrix} \tag{16}$$

Moreover, the Darboux vector of the generator trihedron is

$$U_r = \frac{1}{\Re}(\coth \rho r - k). \tag{17}$$

The natural trihedron consists of the following vectors

$$t = \vec{R}', \; n = \frac{1}{\kappa} t', \; b = n \wedge t \tag{18}$$

where $\kappa = \| t' \|$ is the curvature. The origin of the natural trihedron is a striction point of the normalia. The striction curve is defined as

$$\beta_T(s) = \beta(s) - \mu_T(s) t(s) \tag{19}$$

where

$$\mu_T(s) = \left| \frac{\langle \beta'(s), t'(s) \rangle}{\langle t'(s), t'(s) \rangle} \right| \tag{20}$$

which is the distance from the striction point of the normalia to the striction point of the timelike ruled surface in the positive direction of the central normal vector. Substituting Equation (16) and Equation (11) into Equation (20), we obtain

$$\mu_T(s) = \left| \Re sinh^2 \rho (\Gamma - \Delta \coth \rho) \right|. \tag{21}$$

The first-order angular motion property of the natural trihedron may be determined by

$$\frac{d}{ds}\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \tag{22}$$

where $\tau =< n', b >$ is torsion. The Darboux vector of the natural trihedron is

$$U_t = -\tau t + \kappa b \tag{23}$$

which satisfies $\dfrac{dt}{ds} = U_t \wedge t,\ \dfrac{dn}{ds} = U_t \wedge n,\ \dfrac{db}{ds} = U_t \wedge b$.

Observe that both the Darboux vectors of the generator trihedron and the natural trihedron describe the angular motion of the ruled surface and the central normal surface. From Eq. (15) and Eq. (8) the curvature κ may be written as follows

$$\kappa = \frac{1}{\Re \sinh \rho}. \tag{24}$$

Differentiating Equation (19) and then substituting Equation (8) and Equation (11) into the result, we obtain

$$\beta_T' = \Gamma_T t - \Delta_T b \tag{25}$$

where

$$\Gamma_T = -\mu_T' t, \qquad \Delta_T = \Gamma \cosh \rho - \Delta \sinh \rho . \tag{26}$$

5. RELATIONSHIP BETWEEN THE FRAMES OF REFERENCE

The orientation of the surface frame relative to the tool frame and the generator trihedron is as shown in Figure 2. Let hyperbolic angle between S_b and A timelike vectors be defined as φ, refered as spin angle, we may express results in matrix form as

$$\begin{bmatrix} \mathrm{O} \\ A \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sinh\varphi & \cosh\varphi \\ 0 & -\cosh\varphi & -\sin\varphi \end{bmatrix} \begin{bmatrix} \mathrm{O} \\ S_n \\ S_b \end{bmatrix}. \tag{27}$$

Using Equation (27), we have

$$\begin{bmatrix} \mathrm{O} \\ S_n \\ S_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sinh\varphi & -\cosh\varphi \\ 0 & \cosh\varphi & \sinh\varphi \end{bmatrix} \begin{bmatrix} \mathrm{O} \\ A \\ N \end{bmatrix}. \tag{28}$$

Let the hyperbolic angle between the timelike vectors S_b and k be defined as ϕ. We have

$$\begin{bmatrix} O \\ S_n \\ S_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\phi & \sinh\phi \\ 0 & \sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} r \\ t \\ k \end{bmatrix}. \tag{29}$$

Using Equation (27) and Equation (29), we have

$$\begin{bmatrix} O \\ A \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sinh\Sigma & \cosh\Sigma \\ 0 & -\cosh\Sigma & -\sinh\Sigma \end{bmatrix} \begin{bmatrix} r \\ t \\ k \end{bmatrix} \tag{30}$$

where $\varphi + \phi = \Sigma$. The solution of Equation (30) is obtained by

$$\begin{bmatrix} r \\ t \\ k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sinh\Sigma & -\cosh\Sigma \\ 0 & \cosh\Sigma & \sinh\Sigma \end{bmatrix} \begin{bmatrix} O \\ A \\ N \end{bmatrix} \tag{31}$$

where Σ describes the orientation of the end-effector. Because the surface normal vector is determined at the TCP which is on the directrix, u is zero. Since the orientation vector is coincident with the generator vector, substituting Equation (11), Equation (6) into Equation (5), we have

$$S_n = \frac{-\Delta t + \mu k}{\sqrt{\Delta^2 - \mu^2}}, \; S_b = \frac{-\Delta k + \mu t}{\sqrt{\Delta^2 - \mu^2}} \tag{32}$$

Comparing Equation (32) with Equation (29), we observe that

$$\sinh\phi = \frac{\mu}{\sqrt{\Delta^2 - \mu^2}}, \; \cosh\phi = \frac{-\Delta}{\sqrt{\Delta^2 - \mu^2}}. \tag{33}$$

Substituting Equation (14), Equation (24) into Equation (17) gives

$$U_r = \kappa b \tag{34}$$

which shows that the binormal vector plays the role of the instantaneous axis of rotation for the generator trihedron.

6. DIFFERENTIAL MOTION OF THE TOOL FRAME

In this section, we obtain expressions of the first and second-order positional variation of the TCP. The space curve generated by TCP from Equation (7) is

$$\alpha(s) = \beta(s) + \mu\vec{R}(s) \; . \tag{35}$$

Differentiating Equation (35) with respect to the arc length, using Equation (8) and Equation (11) we obtain the first order positional variation of the TCP which is expressed in the generator trihedron as follows:

$$\alpha'(s) = (\Gamma + \mu'\Re)r - \Delta k + \mu t. \tag{36}$$

Substituting Equation (31) into Equation (36), it gives

$$\alpha'(s) = (\Gamma + \mu'\Re)\mathrm{O} - (\mu\sinh\Sigma + \Delta\cosh\Sigma)A - (\mu\cosh\Sigma + \Delta\sinh\Sigma)N$$

Differentiating Equation (30) and substituting Equation (16) and Equation (31) into the result to determine the first order angular variation of the tool frame, we obtain

$$\frac{d}{ds}\begin{bmatrix} \mathrm{O} \\ A \\ N \end{bmatrix} = \frac{1}{\Re}\begin{bmatrix} 0 & -\sinh\Sigma & -\cosh\Sigma \\ -\sinh\Sigma & 0 & -\Omega\Re \\ \cosh\Sigma & -\Omega\Re & 0 \end{bmatrix}\begin{bmatrix} \mathrm{O} \\ A \\ N \end{bmatrix} \tag{37}$$

where

$$\Omega = \Sigma' + \frac{\coth\rho}{\Re} \; . \tag{38}$$

Moreover the Darboux vector of the generator trihedron is

$$U_O = \Omega\mathrm{O} - \frac{1}{\Re}\cosh\Sigma A - \frac{1}{\Re}\sinh\Sigma N \tag{39}$$

which satisfies $\frac{dO}{ds} = U_O \wedge O,\ \frac{dA}{ds} = U_O \wedge A,\ \frac{dN}{ds} = U_O \wedge N$.

Substituting Equation (31) into Equation (39) gives

$$U_O = \Omega r - \frac{1}{\Re} k. \tag{40}$$

Hence from Equation (38) and Equation (17), we have

$$U_O = \Sigma' r + U_r. \tag{41}$$

Substituting Equation (34) into Equation (41) gives

$$U_O = \Sigma' r + \kappa b. \tag{42}$$

The second order angular variation of the frames may now be obtained by differentiating the darboux vectors. Differentiating Equation (23) and Equation (34) gives

$$U_r' = \kappa' b + \kappa\tau n,\ U_t' = \kappa' t + \tau' b. \tag{43}$$

Differentiating Equation (40), substituting Equation (8) and Equation (38) into the result, the first order derivatives of the tool frame is rewritten as

$$U_O' = \Omega' r + \frac{\Sigma'}{\Re} t. \tag{44}$$

7. EXAMPLE

Figure 3 shows the timelike ruled surface parameterized by

$$\varphi(s,u) = (-\sqrt{2}\cosh s + u\sinh s,\ \ s + \sqrt{2}u, \sqrt{2}\sinh s - u\cosh s).$$

It is easy to see that $\alpha(s) = (-\sqrt{2}\cosh s, s, \sqrt{2}\sinh s)$ is the base curve (timelike). $\vec{R}(s) = (\sinh s,\ \sqrt{2}, -\cosh s)$ is the generator (spacelike). Differentiating α(s) gives

$$\alpha'(s) = (-\sqrt{2}\sinh\ s,\ 1, \sqrt{2}\cos\ hs\ ).$$

The generator trihedron is defined as

$$r = (\sinh s, \sqrt{2}, -\cosh s),$$

$$t = (\cosh s, 0, -\sinh s),$$

$$k = (-\sqrt{2}\sinh s, -1, \sqrt{2}\cosh s).$$

Simple calculation implies that

$$\mu = 0,\ \ \Gamma = 2\sqrt{2},\ \Delta = -3 \text{ and } \ \gamma = \sqrt{2}.$$

The natural trihedron is defined by

$$t = (\cosh s, 0, -\sinh s),$$

$$n = (\sinh s, 0, -\cosh s),$$

$$b = (0, -1, 0)$$

where $\kappa = \sqrt{|\langle t', t'\rangle|} = 1$.

The Darboux vector of generator trihedron is

$$\mathrm{U}_r = (0, -1, 0),\ \mathrm{U}_t = (0, -1, 0).$$

Simple calculation implies that

$$\cosh\phi = \frac{3}{\sqrt{9-\mu^2}},\quad \sinh\phi = \frac{\mu}{\sqrt{9-\mu^2}}.$$

Thus, we obtain

$$\phi' = \frac{3\mu'}{9-\mu^2}, \ \phi'' = \frac{3}{9-\mu^2}(\mu'' + \frac{2\mu(\mu')^2}{9-\mu^2}).$$

Since the spin angle φ is zero, so $\varphi' = 0$, $\Sigma = \phi$, $\Sigma' = \phi'$ and $\Omega = \phi' - \sqrt{2}$. The approach vector and the normal vector are

$$A = \frac{1}{\sqrt{9-\mu^2}}(\mu\cosh s - 3\sqrt{2}\sinh s, -3, 3\sqrt{2}\cosh s - \mu\sinh s)$$

and

$$N = \frac{1}{\sqrt{9-\mu^2}}(\sqrt{2}\mu\sinh s - 3\cosh s, \mu, -\sqrt{2}\cosh s + 3\sinh s),$$

respectively.

The first order positional variation of the TCP may be expressed in the tool frame as

$$\alpha' = (2\sqrt{2} + \mu')O - \frac{-9+\mu^2}{\sqrt{9-\mu^2}}A.$$

Finally the Darboux vector of the tool frame is found as

$$U_O = \Omega O - \frac{3}{\sqrt{9-\mu^2}}A - \frac{\mu}{\sqrt{9-\mu^2}}N.$$

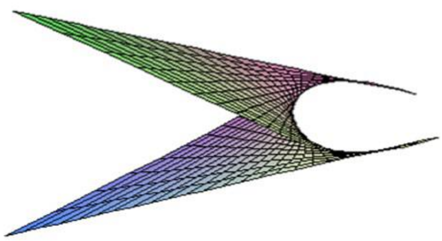

Figure 3: The timelike ruled surface with timelike rulings

8. CONCLUSIONS

In this paper, we have presented the basic mathematical and computational framework for the accurate motion of the end-effector of a robotic device. The paper presents the curvature theory of a general timelike ruled surface. The curvature theory of timelike ruled surfaces is used to determine the differential properties of the motion of a robot end-effector. This provides the properties of the robot end-effector motion in analytical form. The trajectory of a robot end-effector is described by a ruled surface and a spin angle about the rulling of the ruled surface.

REFERENCES

Coken, A. C., Ciftci, U. and Ekici, C. (2008). On parallel timelike ruled surfaces with timelike rulings. *Kuwait Journal of Science & Engineering*. **35**(1A):21-31.

Ekici, C., Unluturk, Y., Dede, M. and Ryuh, B. S. (2008). On motion of Robot End-Effector using the curvature theory of timelike ruled surfaces with timelike rulings. *Hindawi Publishing Corporation Mathematical Problems in Engineering*. ID 362783, [doi:10.1155/2008/362783].

McCarthy, J. M. (1987). On the scalar and dual formulations of line trajectories. *ASME Journal of Mechanisms, Transmissions and Automation in Design*. **109**:101-106.

McCarthy, J. M. and Roth, B. (1981). The curvature theory of line trajectories in spatial kinematics. *ASME Journal of Mechanical Design*. **103**:718-724.

Ogrenmis, A. O., Balgetir, H. and Ergüt, M. (2006). On the ruled surfaces in Minkowski 3-space. *Journal of Zhejiang University: Science A*, **7**(3):326-329.

O'Neill, B. (1983). *Semi-Riemannian geometry with applications to relativity*, vol. 103 of Pure and Applied Mathematics. London, UK: Academic Press.

Paul, R. (1979). Manipulator Cartesian path control. *IEEE Transactions on Systems, Man and Cybernetics.* **9** (1979): 702-711.

Ryuh, B. S. (1989). *Robot trajectory planing using the curvature theory of ruled surfaces*. Doctoral dissertion. West Lafayette, Ind, USA: Purdue University.

Ryuh, B. S. and Pennock, G. R. (1988). Accurate motion of robot end-effector using the curvature theory of ruled surfaces. *Journal of Mechanisms, Transmissions, and Automation in Design*. **110**(4):383-387.

Ryuh, B. S., Lee, K. M. and Moon, M. J. (2006). A Study on the dual curvature theory of a ruled Surface for the precision control of a robot trajectory. *12th International Conference Robotics and Applications*, Hawaii, USA, p.202-207. ISBN hardcopy: 0-88986-595-7.

Turgut, A., Hacısalihoglu, H. H. (1998). Timelike ruled surfaces in the Minkowski 3-space-II. *Turkish Journal of Mathematics*. **22**:33-46.

Turgut, A., Hacısalihoglu, H.H. (1997). Timelike ruled surfaces in the Minkowski 3-space. *Far East Journal of Mathematical Sciences*. **5**(1):83-90.

Wang, D., Liu, J. and Xiao, D. (1997). A unified curvature theory in kinematic geometry of mechanism. *Science in China (Series E)*. **41**(2):196-202.